THE MATRIX TYPE OF PURELY INFINITE SIMPLE LEAVITT PATH ALGEBRAS

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Abstract. Let R denote the purely infinite simple unital Leavitt path algebra $L(E)$. We completely determine the pairs of positive integers (c,d) for which there is an isomorphism of matrix rings $\mathrm{M}_c(R) \cong \mathrm{M}_d(R)$, in terms of the order of $[1_R]$ in the Grothendieck group $K_0(R)$.

For a row-finite directed graph E and field k, the Leavitt path algebra $L_k(E)$ has been defined in [1] and [9], and further investigated in numerous subsequent articles. Purely infinite simple rings were introduced in [8]; the purely infinite simple Leavitt path algebras were explicitly described in [2]. All terminology used in this article can be found in these four references. We denote $L_k(E)$ simply by $L(E)$ throughout.

In this short note we present necessary and sufficient conditions for the existence of a ring isomorphism between the matrix rings $\mathrm{M}_c(L(E))$ and $\mathrm{M}_d(L(E))$ (thereby yielding the so-called *Matrix Type* of $L(E)$), whenever $L(E)$ is both purely infinite simple and unital. ($L(E)$ is unital precisely when the graph E is finite.) The sufficiency of these conditions utilizes the deep "algebraic Kirchberg Phillips Theorem" [7, Theorem 2.5] for Leavitt path algebras: If $L(E)$ and $L(F)$ are Morita equivalent purely infinite simple unital rings, and there exists an isomorphism $\varphi : K_0(L(E)) \to K_0(L(F))$ for which $\varphi([1_{L(E)}]) = [1_{L(F)}]$, then $L(E) \cong L(F)$.

The following result is well-known, but we prove it here for completeness.

2000 *Mathematics Subject Classification.* Primary 16S50, Secondary 16E20.
Key words and phrases. Leavitt path algebra, isomorphism of matrix rings, K-theory.
The first author is partially supported by the U.S. National Security Agency under grant number H89230-09-1-0066.

Lemma 1. *Let G be a finitely generated abelian group (written additively). Let $x \in G$ be an element of finite order n, and let $c, d \in \mathbb{N}$. There exists an automorphism $\varphi : G \to G$ with $\varphi(cx) = dx$ if and only if $\gcd(c, n) = \gcd(d, n)$.*

Proof. ($\Leftarrow$) Since G is finitely generated, $G \cong \mathbb{Z}_{{p_1}^{k_1}} \oplus \cdots \oplus \mathbb{Z}_{{p_s}^{k_s}} \oplus \mathbb{Z}^t$ for some nonnegative integers s, t, (not necessarily distinct) primes p_i ($1 \leq i \leq s$), and $k_i \in \mathbb{N}$. Since x has finite order, we have $x = (x_1, \ldots, x_s, 0, \ldots, 0)$ with $x_i \in \mathbb{Z}_{{p_1}^{k_1}}$. Let m_i denote $\mathrm{ord}(x_i)$. Then $m_i | n$, so we have $\gcd(c, m_i) = \gcd(\gcd(c, n), m_i)$, which by hypothesis equals $\gcd(\gcd(d, n), m_i)$, which (again using $m_i | n$) equals $\gcd(d, m_i)$. Consequently, $\mathrm{ord}(cx_i) = m_i / \gcd(c, m_i) = m_i / \gcd(d, m_i) = \mathrm{ord}(dx_i)$. Since $\mathbb{Z}_{{p_i}^{k_i}}$ is cyclic, and cx_i and dx_i have the same order in $\mathbb{Z}_{{p_i}^{k_i}}$, there exists an automorphism φ_i of $\mathbb{Z}_{{p_i}^{k_i}}$ with $\varphi_i(cx_i) = dx_i$. Now define $\varphi = \varphi_1 \oplus \cdots \oplus \varphi_s \oplus \mathrm{id}_{\mathbb{Z}} \oplus \cdots \oplus \mathrm{id}_{\mathbb{Z}} \in \mathrm{Aut}(G)$; then clearly $\varphi(cx) = dx$, as desired.

($\Rightarrow$) Conversely, suppose $cx \overset{\varphi}{\mapsto} dx$ for some automorphism φ of G. Then $n / \gcd(c, n) = \mathrm{ord}(cx) = \mathrm{ord}(dx) = n / \gcd(d, n)$, so $\gcd(c, n) = \gcd(d, n)$. □

Our first of two main results generalizes to all purely infinite simple unital Leavitt path algebras $L(E)$ a result known previously for the Leavitt algebras L_q. We note for later use that when E is finite, the semigroup $\mathcal{V}^*(L(E))$, and therefore the group $K_0(L(E))$, is finitely generated by [9, Theorem 3.5].

Theorem 2. *Let E be a graph for which $L(E)$ is purely infinite simple unital. Suppose $[1_{L(E)}] \in K_0(L(E))$ has finite order n. Then, for any $c, d \in \mathbb{N}$, there exists an isomorphism of matrix rings $\mathrm{M}_c(L(E)) \cong \mathrm{M}_d(L(E))$ if and only if $\gcd(c, n) = \gcd(d, n)$.*

Proof. ($\Rightarrow$) Because $K_0(L(E))$ is finitely generated, Lemma 1 ensures that there exists $\varphi \in \mathrm{Aut}(K_0(L(E)))$ having $\varphi(c[1_{L(E)}]) = d[1_{L(E)}]$.

Let $m \in \mathbb{N}$. By the standard Morita equivalence $\Psi : \mathrm{M}_m(L(E)) \sim L(E)$ we have the induced isomorphism

$$\psi_m : K_0(\mathrm{M}_m(L(E))) \to K_0(L(E))$$

for which $\psi_m([1_{\mathrm{M}_m(L(E))}]) = m[1_{L(E)}]$. By [6, Proposition 9.3] there exists a graph M_mE for which $L(M_mE) \cong \mathrm{M}_m(L(E))$. Specifically, this yields an isomorphism

$$\rho_m : K_0(L(M_mE)) \to K_0(\mathrm{M}_m(L(E)))$$

for which $\rho_m([1_{L(M_mE)}]) = [1_{\mathrm{M}_m(L(E))}]$. Now the composition

$$\kappa = \rho_d^{-1} \circ \psi_d^{-1} \circ \varphi \circ \psi_c \circ \rho_c$$

is an isomorphism from $K_0(L(M_cE))$ to $K_0(L(M_dE))$ for which $\kappa([1_{L(M_cE)}]) = [1_{L(M_dE)}]$.

Since $L(M_cE)$ and $L(M_dE)$ are both Morita equivalent to $L(E)$ and therefore to each other, the existence of the isomorphism κ having the indicated property allows us to apply the aforementioned algebraic Kirchberg Phillips Theorem [7, Theorem 2.5], from which we conclude that there is an isomorphism $L(M_cE) \cong L(M_dE)$, which yields $\mathrm{M}_c(L(E)) \cong \mathrm{M}_d(L(E))$ as desired.

($\Leftarrow$) Conversely, suppose $\gcd(c,n) \neq \gcd(d,n)$. In this case there cannot be a ring isomorphism from $\mathrm{M}_c(L(E))$ to $\mathrm{M}_d(L(E))$, as otherwise, by contradiction, if such exists then (by standard ring theory, see. e.g. [11, p. 5]) there would exist an isomorphism

$$\tau : K_0(\mathrm{M}_c(L(E))) \to K_0(\mathrm{M}_d(L(E)))$$

for which $\tau([1_{\mathrm{M}_c(L(E))}]) = [1_{\mathrm{M}_d(L(E))}]$. But then $\varphi = \psi_d \circ \tau \circ \psi_c^{-1}$ (with ψ_m as above) would be an automorphism of $G = K_0(L(E))$ for which $\varphi(c[1_{L(E)}]) = d[1_{L(E)}]$, which is impossible by Lemma 1. $\square$

In fact, the proof given above yields that the converse direction of Theorem 2 holds for all rings R for which $[1_R]$ has finite order in $K_0(R)$.

We note that Theorem 2 generalizes [3, Theorem 4.14] and [3, Theorem 5.2] (as well as [4, Theorem 5.9]) from the (purely infinite simple) Leavitt algebras L_q to all purely infinite simple unital Leavitt path algebras for which $[1_{L(E)}]$ has finite order in $K_0(L(E))$, since the

order of $[1_{L_q}]$ in $K_0(L_q)$ is $q-1$. In the related article [5] we will show that the indicated isomorphisms between matrix rings can be explicitly described.

To complete the determination of the Matrix Type of all purely infinite simple unital Leavitt path algebras, we now consider the case where $[1_{L(E)}]$ has infinite order in $K_0(L(E))$.

Lemma 3. *Let G be a finitely generated abelian group. If there exists $m, n \in \mathbb{N}$, $\sigma \in \mathrm{Aut}(G)$, and $x \in G$ of infinite order such that $n\sigma(x) = mx$, then $n = \pm m$.*

Proof. Since G is finitely generated, $G \cong H \oplus \mathbb{Z}^t$ for some $t \in \mathbb{N}$, with H finite. By hypothesis, x has nonzero component $\hat{x}$ in $\mathbb{Z}^t$. We easily get that $\mathrm{Aut}(G) = \mathrm{Aut}(H) \oplus \mathrm{Aut}(\mathbb{Z}^t)$. Since $\mathrm{Aut}(\mathbb{Z}^t) = GL(t, \mathbb{Z})$, since $\sigma(\hat{x}) = \frac{m}{n}\hat{x}$ by hypothesis, and since the only rational eigenvalues of an invertible integer-valued matrix are 1 and -1, we have $\frac{m}{n} = \pm 1$, which gives the result. $\square$

Following terminology introduced by P. Vamos, we say that a ring R has *Invariant Matrix Number* in case $\mathrm{M}_i(R) \not\cong \mathrm{M}_j(R)$ for every pair of positive integers $i \neq j$.

Proposition 4. *Let R be a unital ring for which the order of $[1_R]$ in $K_0(R)$ is infinite, and for which $K_0(R)$ is a finitely generated group. Then R has Invariant Matrix Number.*

In particular, if E is finite, and $[1_{L(E)}]$ has infinite order in $K_0(L(E))$, then $L(E)$ has Invariant Matrix Number.

Proof. Let $m, n \in \mathbb{N}$ and suppose $\mathrm{M}_m(R) \cong \mathrm{M}_n(R)$. Then, as noted above, there exists an isomorphism $\tau : K_0(\mathrm{M}_m(R)) \to K_0(\mathrm{M}_n(R))$ for which $\tau([1_{\mathrm{M}_m(R)}]) = [1_{\mathrm{M}_n(R)}]$. In addition, as noted previously, for any $a \in \mathbb{N}$, using the standard Morita equivalence between R and $\mathrm{M}_a(R)$ we get an isomorphism $\psi_a : K_0(\mathrm{M}_a(R)) \to K_0(R)$ for which $\psi_a([1_{\mathrm{M}_a(R)}]) = a[1_R]$. Then the composition

$$\sigma = \psi_n \circ \tau \circ \psi_m^{-1} : K_0(R) \to K_0(R)$$

is an automorphism of $K_0(R)$ for which $m\sigma([1_R]) = n[1_R]$. By Lemma 3 and because $[1_R]$ has infinite order, $m = n$.

The result applies immediately to the indicated rings of the form $L(E)$ since, as noted above, for these rings $K_0(L(E))$ is a finitely generated abelian group. $\square$

The requirement that $K_0(R)$ be finitely generated cannot be removed from Proposition 4. We thank E. Pardo for providing the following example and subsequent remarks.

Example 5. Let R be any unital ring. Consider the ring $S = \varinjlim(\mathrm{M}_n(R), f_{m,n})$, where the connecting maps $f_{m,n} : \mathrm{M}_m(R) \to \mathrm{M}_n(R)$ are defined when m divides n, and are the classical block diagonal maps. It is well known (and not hard to show) that $S \cong \mathrm{M}_n(S)$ for every $n \in \mathbb{N}$. (That is, S has *Single Matrix Number.*)

Suppose also that R has the property that $[1_R]$ has infinite order in $K_0(R)$. Since K_0 is a continuous functor, we have $K_0(S) \cong \varinjlim K_0(\mathrm{M}_n(R))$. As utilized above, we have $(K_0(\mathrm{M}_n(R)), [I_n]) \cong (K_0(R), n[1_R])$. Since $K_0(f_{m,n})([I_m]) = [I_n]$, we conclude that the order of $[1_S]$ in $K_0(S)$ is infinite as well.

Thus for R any unital ring for which $[1_R]$ has infinite order in $K_0(R)$, the ring $S = \varinjlim(\mathrm{M}_n(R), f_{m,n})$ has the property that $[1_S]$ is of infinite order in $K_0(S)$, and for which S does not have Invariant Matrix Number, as desired.

Our interest here is in purely infinite simple rings R, so one might ask whether the finitely generated hypothesis can be dropped from Proposition 4 in case R has this additional property. But even in this case the finitely generated hypothesis on $K_0(R)$ is needed, since if one starts with R purely infinite simple in the previous Example, then S can easily be shown to be purely infinite simple as well.

For any unital ring R, if we construct $S = \varinjlim \mathrm{M}_n(R)$ as in the Example, then it is well known that

$$K_0(S) \cong \mathbb{Q} \otimes_{\mathbb{Z}} K_0(R).$$

(To establish this, for each $n \in \mathbb{N}$ and $[A] \in K_0(M_n(R))$ define $\kappa_n : K_0(M_n(R)) \to \mathbb{Q} \otimes_{\mathbb{Z}} K_0(R)$ by setting $\kappa_n([A]) = \frac{1}{n} \otimes \psi_n([A])$. One then verifies that the maps κ_n are consistent

with the maps $K_0(f_{m,n})$, which then yields a homomorphism from $K_0(S) \cong \varinjlim K_0(\mathrm{M}_n(R))$ to $\mathbb{Q} \otimes_{\mathbb{Z}} K_0(R)$. That this map is an isomorphism is easily shown by constructing the appropriate inverse map.) In particular, for every $m, n \in \mathbb{N}$, we may define $\sigma \in Aut(K_0(S))$ as the linear extension of $q \otimes t \mapsto \frac{m}{n} q \otimes t$ for $q \in \mathbb{Q},\ t \in K_0(R)$. Then $\sigma \in \mathrm{Aut}(K_0(S))$, and for every $x \in K_0(S)$ we have $n\sigma(x) = mx$. (Compare this to the hypotheses of Lemma 3.) This observation is what accounts for the difference in the Matrix Type of the ring S given here (i.e., Single Matrix Number), as compared to the Invariant Matrix Number property of Leavitt path algebras of finite graphs E for which $[1_{L(E)}]$ has infinite order in $K_0(L(E))$.

References

[1] G. Abrams, G. Aranda Pino, The Leavitt path algebra of a graph, *J. Algebra* **293** (2005), 319-334.

[2] G. Abrams, G. Aranda Pino, Purely infinite simple Leavitt path algebras, *J. Pure Appl. Algebra*, **207** (2006), 553-563.

[3] G. Abrams, P. N. Ánh, E. Pardo, Isomorphisms between Leavitt algebras and their matrices, *J. reine Angew. Math.* **624** (2008), 103-132.

[4] G. Abrams, P. N. Ánh, A. Louly, E. Pardo, The classification question for Leavitt path algebras, *J. Algebra* **320** (2008), 1983–2026.

[5] G. Abrams, C. Smith, Explicit isomorphisms between Leavitt path algebras, in preparation.

[6] G. Abrams, M. Tomforde, Isomorphism and Morita equivalence of graph algebras, submitted. arXiv:0810.2569v1

[7] G. Abrams, A. Louly, E. Pardo, C. Smith, Flow invariants in the classification of Leavitt path algebras, submitted. arXiv:0812.0553v3

[8] P.Ara, K. Goodearl, E. Pardo, K_0 of purely infinite simple regular rings, *K-Theory* **26** (2002), 69-100.

[9] P. Ara, M.A. Moreno, E. Pardo, Nonstable K-Theory for graph algebras, *Algebr. Represent. Theory* **10** (2007), 157-178.

[10] G. Aranda Pino, E. Pardo, M. Siles Molina, Exchange Leavitt path algebras and stable rank, *J. Algebra* **305** (2006), 912–936.

[11] J. Rosenberg, Algebraic K-Theory and its Applications, Graduate Texts in Mathematics 147, Springer-Verlag, New York, 1994, x + 392pp.

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